Longitude shift versus Eccentricity for a Geostationary spacecraft

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1 Purpose of the analysis

This short note analyses the Greenwich longitude shift for Geostationary orbits having eccentricity.

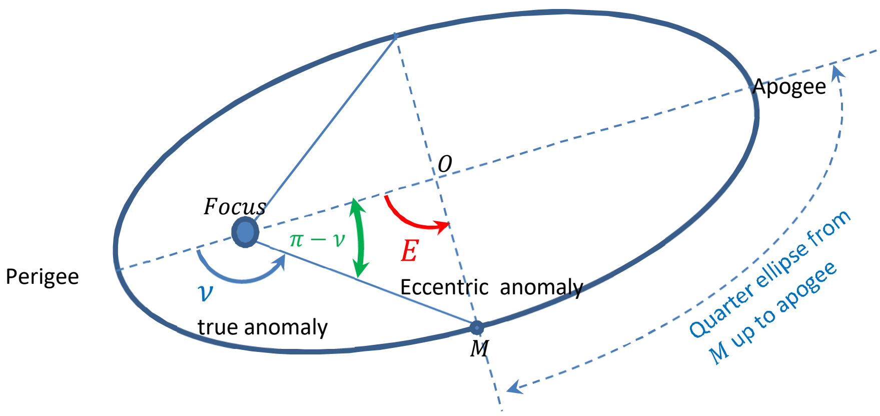
2 Generalities

The orbital period is $T = 2\pi\sqrt{\frac{a^3}{\mu}}$, it is not dependent on the eccentricity, it depends only on a the semi-major axis. However, the time evolution along the orbit depends on the eccentricity.

2.1 Angles considered

The eccentric anomaly is the very well-known angle E taken into account when writing an ellipse parametrically in the frame major axis and minor axis centered in O (b being the semi-minor axis):

$$x = a.\cos(E),$$
$$y = b.\sin(E).$$



The half of the ellipse, as shown with point M on the figure above, is of course characterized by an eccentric anomaly E of $\frac{\pi}{2}$ i.e. 90°.

This point M is also given for a true anomaly ν from the perigee with respect to the $Focus$. The relation between true anomaly and eccentric anomaly is well-known: $\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan(\frac{E}{2})$.

3 Angular velocities and problem reduction

The angular velocity $\frac{d\nu}{dt}$ of the spacecraft for the half ellipse around perigee is higher that the angular velocity for the half ellipse around apogee.
Hence the spacecraft spend more time on the apogee half ellipse than on the other half ellipse.
The sum being equal to the orbital period T, it is sufficient to consider the effects of the eccentricity only on the half orbit around apogee.
And because the evolutions are symmetric before and after apogee, it is thus sufficient to consider the effects of the eccentricity only on the quarter ellipse from M up to apogee.
The longitude shift equation comes from the difference between two angles:

- The angle the spacecraft travel in some time,
- The angle the Greenwich meridian of $Focus$ Earth rotates in the same time.

During the time Δt spent by the spacecraft on the quarter ellipse, the $Focus$ Earth Greenwich meridian rotates by an angle given by $\frac{2\pi}{T_{Earth}}\Delta t$ with T_{Earth} the sidereal period of the Earth spin.
During the same time Δt the spacecraft travel only by an angle of $\pi - \nu$ with respect to the $Focus$.
Thus, the difference between those two angles give the longitude shift.
Hence two things must be assessed: Δt given by the time equation and ν given by the eccentric to true anomaly equation.

4 Longitude shift equation

4.1 Time equation

When the spacecraft move from the half ellipse (point M) up to apogee, the eccentric anomaly vary from $E_1 = \frac{\pi}{2}$ to $E_2 = \pi$ while the true anomaly move from ν to π.
The time Δt needed for the spacecraft to travel from E_1 to E_2 is given by the Kepler time equation applied to both points, T being the orbital period, e the eccentricity: $t(E) = \frac{T}{2\pi}(E - e.\sin(E))$
Hence the duration is $\Delta t = \frac{T}{2\pi}\left(\pi - \frac{\pi}{2} + e\right)$.

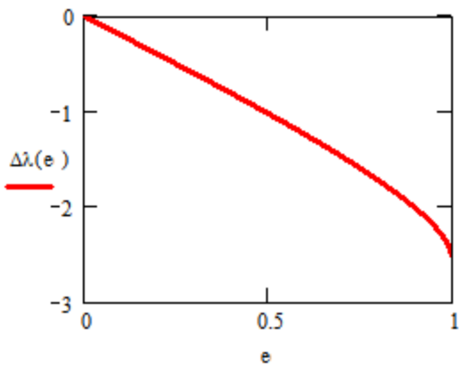
4.2 Eccentric to true anomaly equation

This relation is well-known, solved for ν it gives: $\nu = 2.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$. For the point M in the middle of the ellipse, $E = \frac{\pi}{2}$, so $\frac{E}{2} = \frac{\pi}{4}$ and $\tan\frac{\pi}{4} = 1$ so it remains: $\nu = 2.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\right)$.

4.3 Synthesis

The longitude of the spacecraft with respect to the rotating Earth is given by the difference between the variation of true anomaly of the spacecraft $\pi - \nu$ and the angle of the Earth rotation during for the same time Δt .
Hence it is straightforward that $\Delta\lambda = (\pi - \nu) - \frac{2\pi}{T_{Earth}}\Delta t$.
Because for a geostationary spacecraft the period $T = T_{Earth}$, that gives: $\Delta\lambda = (\pi - \nu) - \left(\pi - \frac{\pi}{2} + e\right)$ $\quad \Delta\lambda = -\nu + \frac{\pi}{2} - e$

$$\Delta\lambda = -2.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\right) + \frac{\pi}{2} - e \quad \textit{in radian}$$



A plot of $\Delta\lambda$ *in radian* versus e is shown at right [R 1]. The relation $\Delta\lambda$ versus e is very linear up to $e = 0.5$ and more. As detailed in the next §, the longitude shift over a whole orbit is $\pm\ \Delta\lambda$ wrt to the mean longitude.

5 Longitude shift equation for small eccentricities

The development of $\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\right)$ versus small values of e is given by [R 1]:

$$\operatorname{atan}\left(\sqrt{\frac{1+e}{1-e}}\right) = \frac{1}{4}\cdot\pi + \frac{1}{2}\cdot e + \frac{1}{12}\cdot e^3 + \frac{3}{80}\cdot e^5 + O\left(e^6\right)$$

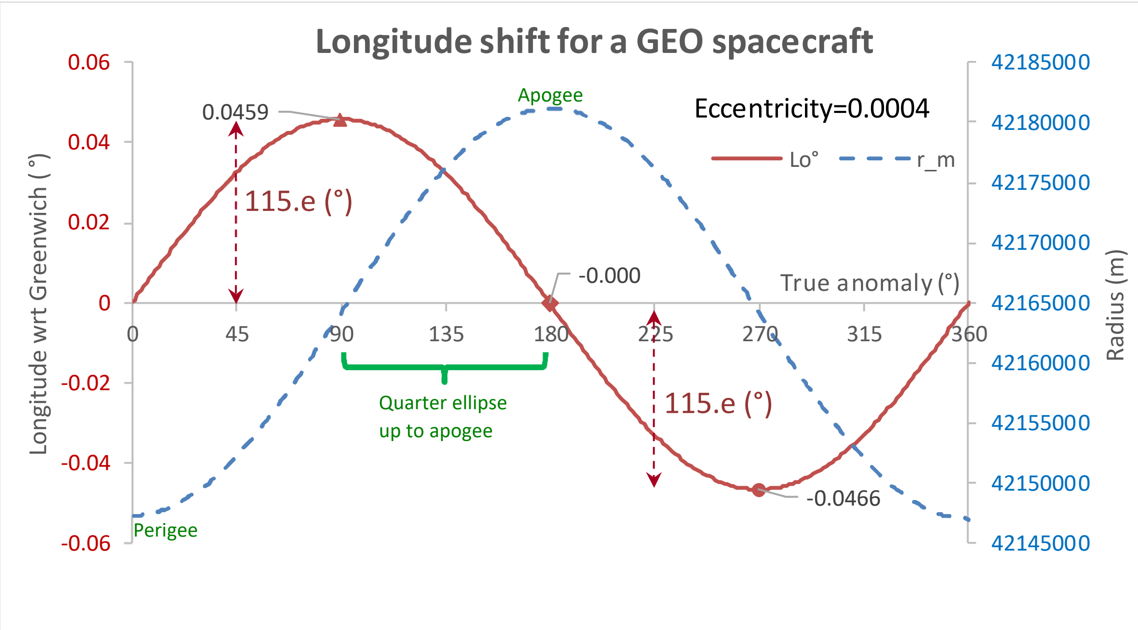
so it remains: $\Delta\lambda = -2\left(\frac{\pi}{4}+\frac{e}{2}\right)+\frac{\pi}{2}-e$. *Note: it is remarkable to see that each of the two contributions to the longitude shift ($\pi - \nu$ and Δt) lead to half of the total value.* Eventually one gets the value in radian:

- $\Delta\lambda = -2e$ (for the quarter of ellipse M to Apogee),
- $\Delta\lambda = -2e$ (for the quarter of ellipse Apogee to symmetrical of M),
- Hence min-max amplitude for half ellipse around the apogee is $2\Delta\lambda = -4e$ radian,
- And min-max amplitude for the half ellipse around the perigee is $2\Delta\lambda = +4e$ radian.

For small e , the longitude shift along one orbit is $\Delta\lambda = \pm 2e$ *in radian* [R 2], that is also $\Delta\lambda = \pm\frac{360}{\pi}e$ degrees $\Delta\lambda = \pm 115\, e$ *in degrees* .

The plot below [R 3] confirms the validity of those equations for $e = 0.0004$ with a mean longitude of 0.000° and the mean-max amplitude of longitude shift of 0.0459 ° which is 115 x 0.0004 ° .

In addition, for small e, one gets the longitude wrt Greenwich $\lambda \approx \lambda_0 + 115\, e \sin\nu$ *in degrees*.



6 References:

[R 1] Mathcad tool
[R 2] Cours de technologie spatiale : Techniques et Technologies des Véhicules Spatiaux, CNES, 1995
[R 3] KopooS, TriaXOrbitaL tool 1989-2021

La mécanique orbitale est une discipline étrange... La première fois que vous la découvrez, vous ne comprenez rien... La deuxième fois, vous pensez que vous comprenez, sauf un ou deux points... La troisième fois, vous savez que vous ne comprenez plus rien, mais à ce niveau vous êtes tellement habitué que ça ne vous dérange plus. attribué à Arnold Sommerfeld pour la thermodynamique, vers 1940
